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Nonlinear electrohydrodynamic Rayleigh–Taylor instability. Part 1. A perpendicular field in the absence of surface charges

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Nonlinear electrohydrodynamic Rayleigh–Taylor instability is investigated. A charge-free surface separating two semi-infinite dielectric fluids influenced by a normal electric field is subjected to nonlinear deformations. We use the method of multiple-scale perturbations in order to obtain uniformly valid expansions near the cutoff wavenumber separating stable from unstable flows. We obtain two nonlinear Schrödinger equations by means of which we can deduce the cutoff wavenumber and analyse the stability of the system. It is found that if a finite-amplitude wave exists then its small modulation is stable. We also obtain the surface elevation for such waves. The electric field plays a dual role in the stability criterion and the dielectric constant plays a distinctive role in this analysis. If the dielectric constant of the upper fluid is smaller than that of the lower fluid the field has a destabilizing effect for large wavenumbers. For relatively smaller wavenumbers the electric field stabilizes considerable parts of the first and second subharmonic regions in the stability diagrams; a result which is in contrast with the linear theory. If the dielectric constant of the upper fluid is larger than that of the lower fluid, then the field is stabilizing for larger values of the wavenumber K' when ρ is small (ρ is the density ratio) and destabilizing for smaller values of K'.

1. Introduction

It was reported experimentally by Gross & Porter (1966) that when a thermally and gravitationally stable stratification of a fluid is subjected to an electric field, instability occurs. This instability was explained by the suggestion that when a non-uniform field is applied to an inhomogeneous fluid having variable dielectric constant, regions of lower dielectric constant will experience a force directed towards regions of lower field strength. Thus instability may arise due to variation of the dielectric constant because of heating or any source of inhomogeneity. This experiment was followed by several attempts to explain this phenomenon under various conditions using the Rayleigh–Bénard model, such as those by Roberts (1969), Takashima & Aldridge (1976), Bradley (1978), Castellanos & Velarde (1981). It is found that the temperature dependence of the dielectric constant has no significant effects on the fluid layer, and stabilizing or destabilizing effects of the electric field depend on whether the conductivity is a linear function of temperature or a quadratic function. The coupling of dielectric-constant variation and unipolar injection drastically alters the stability properties of the dielectric layer. Lacroix, Atten & Hopfinger (1975) claimed that there is no analogy with the Bénard problem because the instability is nonlinear. The electrohydrodynamic body force is nonlinear and the phenomenon cannot be accurately described by the linear theory. Studies in nonlinear electrohydrodynamic bulk instability then followed by Atten & Lacroix (1979) and Warraker & Richardson (1981).

The abovementioned effect of the variation of the dielectric constant on bulk electrohydrodynamic stability leads one to expect similar behaviour in surface electrohydrodynamic stability. We feel that the stability criterion for a surface separating two uncharged homogeneous dielectrics should be affected by the jump in the values of the dielectric constants and electric field across the surface. Such effects can appear through nonlinear analysis, since the linear theory did not predict them. In spite of the vanishing of the electrohydrodynamic body force in the bulk of the fluids, the integration of the body force across the surface will contribute to the stability criterion. There is experimental evidence of the nonlinear behaviour of electrohydrodynamic surface stability. Taylor (1969) reported that oscillations may be observed near the linear cutoff wavenumber separating stable and unstable disturbances. Mel'nikov & Meshkov (1981) suggested a nonlinear model to explain the charged dimples of the surface of liquid helium observed by Wanner & Leiderer (1979).

We are prompted by these considerations to study the nonlinear electrohydrodynamic stability of a Rayleigh–Taylor surface separating two dielectric fluids stressed by a normal electric field. In this paper we shall only study the case where there are no surface charges present on the interface. We shall be concerned with the following two main points.

(i) We shall examine the effect of the dielectric constant on the stability criterion. In the linear theory, the stability criterion is governed by the dispersion equation (Melcher 1963; Mohamed & Nayyar 1973)

$$\omega^2 = \frac{K'}{\rho^{(2)}+\rho^{(1)}}\left[TK'^2 - g'(\rho^{(2)}-\rho^{(1)}) - \frac{K'E_0^{(2)}E_0^{(1)}\,(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})^2}{\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)}}\right],$$

where $\tilde{\epsilon}^{(1)}E_0^{(1)} = \tilde{\epsilon}^{(2)}E_0^{(2)}$, g' is the acceleration due to gravity, K' is the wavenumber of the disturbance, T is the surface tension, $\rho^{(2),(1)}$ are the upper and lower fluid densities respectively, $\tilde{\epsilon}^{(2),(1)}$ are the upper and lower dielectric constants, $E_0^{(2),(1)}$ are the electric fields and ω is the frequency of the disturbance. The ratio of the dielectric constants has no effect in the sense that it is immaterial which of the fluids has a larger dielectric constant.

(ii) We aim to study the stability of the system in general and particularly near the cutoff wavenumber K'_{cE} obtained from the above relation by equating ω to zero, exploring the possibility of having oscillations near the cutoff wavenumber at positions considered unstable in the linear sense. Moreover, we wish to examine whether the electric field is strictly destabilizing, as predicted by the linear theory, or otherwise.

Very few studies on nonlinear electrohydrodynamic Rayleigh–Taylor instability have been attempted. Melcher (1963) and Michael (1977) studied the nonlinear stability of the interface of a fluid of finite depth stressed by a normal electric field. In their models, there are charges on the interface. They studied conducting fluids, and therefore the effect of the dielectric constants is not accounted for in their analysis. The nonlinear cutoff wavenumbers were not evaluated in their studies.

Kant, Jindia & Malik (1981) investigated the stability of weakly nonlinear waves on the surface of a fluid layer in the presence of an applied electric field by using the derivative expansion method. They also studied conducting fluids, and therefore the effect of the dielectric constants was also not accounted for in their analysis.

Works on pure hydrodynamic Rayleigh–Taylor instabilities, such as those of Taylor (1950), Lewis (1950), Ingraham (1954), Emmons, Chang & Watson (1960), Rajappa & Amaranath (1977), Davey & Stewartson (1974), Nayfeh (1969, 1976), showed that there are variations in the stability conditions in the nonlinear analysis from those of the linear theory. In order to evaluate the nonlinear cutoff wavenumber, care must be taken in choosing the perturbation technique for the problem at hand. The various perturbation expansions should be uniformly valid near the linear cutoff wavenumber. The method of multiple-scale perturbations was used successfully by Hasimoto & Ono (1972) for fluid of finite depth, and by Nayfeh (1976) for fluids of infinite depth, which is relevant to the present case. The latter derived two nonlinear Schrödinger equations describing the wave propagation on the surface. One of these equations is valid near the cutoff wavenumber.

2. Formulation of the problem

Consider two semi-infinite dielectric inviscid fluids separated by the plane $y = 0$. The upper and lower densities of the fluids are $\rho^{(2)}$, $\rho^{(1)}$ respectively. Both of the fluids are subject to a constant electric field in the y-direction ($E_0^{(2)}$, $E_0^{(1)}$ respectively).

We shall assume that there are no surface charges at the surface of separation in the equilibrium state, and therefore the electric displacement is continuous at the interface, i.e.

$$\tilde{\epsilon}^{(1)}E_0^{(1)} = \tilde{\epsilon}^{(2)}E_0^{(2)}.$$

In our analysis the various quantities are non-dimensionalized using the characteristic length $L = (T/\rho^{(1)}g')^{\frac{1}{2}}$ and the characteristic time $(L/g')^{\frac{1}{2}}$, where T is the surface tension and g' is the acceleration due to gravity acting in the negative y-direction. The superscripts (1) and (2) refer to quantities in the lower fluid and upper fluid respectively. The velocity potential ϕ satisfies the equation

$$\frac{\partial^2\phi^{(2),(1)}}{\partial x^2} + \frac{\partial^2\phi^{(2),(1)}}{\partial y^2} = 0, \tag{1}$$

where

$$\mathbf{V}^{(2),(1)} = \nabla\phi^{(2),(1)}.$$

The solutions for ϕ have to satisfy, at large distance, the conditions

$$|\nabla\phi^{(2)}| \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \tag{2}$$

$$|\nabla\phi^{(1)}| \rightarrow 0 \quad \text{as} \quad y \rightarrow -\infty. \tag{3}$$

The condition that the interface ($y = \xi(x,t)$) is moving with the fluid leads to

$$\xi_t - \phi_y^{(2),(1)} + \phi_x^{(2),(1)}\,\xi_x = 0 \quad \text{at} \quad y = \xi. \tag{4}$$

We also assume that the quasi-static approximation is valid and we introduce the electrostatic potentials $\psi^{(1)}$ and $\psi^{(2)}$ such that

$$\mathbf{E}^{(2),(1)} = E_0^{(2),(1)}\mathbf{e}_y - \nabla\psi^{(2),(1)}. \tag{5}$$

Therefore the differential equation satisfied by $\psi^{(2),(1)}$ is Laplace's equation

$$\frac{\partial^2\psi^{(2),(1)}}{\partial x^2} + \frac{\partial^2\psi^{(2),(1)}}{\partial y^2} = 0 \tag{6}$$

along with the boundary conditions that the tangential component of the electric field is continuous at the interface, so that

$$\xi_x[\![\psi_y]\!] + [\![\psi_x]\!] = \xi_x[\![E_0]\!] \quad \text{at} \quad y = \xi, \tag{7}$$

where $\{\!\!\{\ \}\!\!\}$ represents the jump across the interface. Since there are no surface charges at the surface $y = \xi$, the normal electric displacement is continuous at the interface. Therefore

$$\xi_x \{\!\!\{\tilde{\epsilon}\psi_x\}\!\!\} - \{\!\!\{\tilde{\epsilon}\psi_y\}\!\!\} = 0 \quad \text{at} \quad y = \xi. \tag{8}$$

The stress tensor is given by

$$\Pi_{ij} = -\Pi\delta_{ij} + \tilde{\epsilon}E_i E_j - \tfrac{1}{2}\tilde{\epsilon}E_\nu^2 \delta_{ij}, \tag{9}$$

where $\Pi = p - \frac{1}{2}\tilde{\epsilon}E^2$. The normal hydrodynamical stress is balanced by the normal electric stress. The balance condition is then

$$\begin{aligned}
\phi_t^{(1)} - \rho\phi_t^{(2)} &+ (1-\rho)\,\xi + \tfrac{1}{2}(\phi_x^{(1)\,2} - \rho\phi_x^{(2)\,2}) + \tfrac{1}{2}(\phi_y^{(1)\,2} - \rho\phi_y^{(2)\,2}) \\
&= \xi_{xx}(1+\xi_x^2)^{-\frac{3}{2}} - \tfrac{1}{2}\{\!\!\{\tilde{\epsilon}\psi_x^2\}\!\!\} - \{\!\!\{\tilde{\epsilon}E_0\,\psi_y\}\!\!\} + \tfrac{1}{2}\{\!\!\{\tilde{\epsilon}\psi_y^2\}\!\!\} + 2\xi_x\,\{\!\!\{\tilde{\epsilon}E_0\,\psi_x\}\!\!\} \\
&\quad - 2\xi_x\,\{\!\!\{\tilde{\epsilon}\psi_x\,\psi_y\}\!\!\} + \xi_x^2\,\{\!\!\{\tilde{\epsilon}\psi_x^2\}\!\!\} - \xi_x^2\,\{\!\!\{\tilde{\epsilon}E_0^2\}\!\!\} + 2\xi_x^2\,\{\!\!\{\tilde{\epsilon}E_0\,\psi_y\}\!\!\} - \xi_x^2\,\{\!\!\{\tilde{\epsilon}\psi_y^2\}\!\!\} \\
&\quad - 2\xi_x^3\,\{\!\!\{\tilde{\epsilon}E_0\,\psi_x\}\!\!\} + 2\xi_x^3\,\{\!\!\{\tilde{\epsilon}\psi_x\psi_y\}\!\!\} + O(\xi_x^4) \quad \text{at} \quad y = \xi,
\end{aligned} \tag{10}$$

where

$$\rho = \rho^{(2)}/\rho^{(1)}.$$

The set of equations (1), (4), (6)–(8) and (10) will be solved using the method of multiple scales (Nayfeh 1973, 1976). We expand the various variables in ascending powers in terms of a small dimensionless parameter ϵ characterizing the steepness ratio of the wave. The independent variables x, t are scaled in a like manner,

$$X_n = \epsilon^n x, \quad T_n = \epsilon^n t, \tag{11}$$

and the variables may be expanded as

$$\xi(x,t) = \sum_{n=1}^{3} \epsilon^n \xi_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\epsilon^4), \tag{12}$$

$$\psi^{(2),(1)}(x,y,t) = \sum_{n=1}^{3} \epsilon^n \psi_n^{(2),(1)}(X_0, X_1, X_2, y, T_0, T_1, T_2) + O(\epsilon^4), \tag{13}$$

$$\phi^{(2),(1)}(x,y,t) = \sum_{n=1}^{3} \epsilon^n \phi_n^{(2),(1)}(X_0, X_1, X_2, y, T_0, T_1, T_2) + O(\epsilon^4). \tag{14}$$

Substituting from (12)–(14) into (1), (4), (6)–(8) and (10) and equating coefficients of equal powers of ϵ, we get three sets of equations of order ϵ, ϵ^2, ϵ^3. These sets of equations and their solutions are given in the appendix.

The solutions of the first-order problem are valid provided that ω_0^2 satisfies the relation

$$\omega_0^2 = \frac{K'}{1+\rho}(1 - \rho - K'\alpha_E + K'^2), \tag{15}$$

where

$$\alpha_E = \frac{E_0^{(2)}E_0^{(1)}(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2}{\epsilon^{(2)} + \epsilon^{(1)}}.$$

Equation (15) is the same linear dispersion equation as that discussed in §1. It is clear that the electric field is strictly destabilizing in the linear sense. The dispersion relation does not depend on the sign of $\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)}$. The stability of the interface depends on whether the wavenumber K' is larger or smaller than K'_{cE}, where

$$K'_{cE} = (\rho - 1)^{\frac{1}{2}}\,[\sinh\tilde{\theta}_E + \cosh\tilde{\theta}_E], \tag{16}$$

$$\sinh\tilde{\theta}_E = \frac{\alpha_E}{2(\rho-1)^{\frac{1}{2}}}. \tag{17}$$

The critical wavenumber K'_{cE} is the linear electrohydrodynamic cutoff wavenumber separating stable from unstable disturbances. We shall be concerned with the case $K' \geqslant K'_{cE}$.

The second-order problem yields the solvability condition

$$\left[2K' + (1+\rho)\frac{\omega_0^2}{K'^2} - \tilde{\alpha}_E\right]\frac{\partial D}{\partial X_1} + 2(1+\rho)\frac{\omega_0}{K'}\frac{\partial D}{\partial T_1} = 0, \tag{18}$$

where $D(X_1, X_2, T_1, T_2)$ is the amplitude of the first-order wave (see equation (A 7)), and for the third-order problem we have the condition

$$i\left[-E_0^{(1)}E_0^{(2)}\frac{(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})^2}{\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)}} + 2K' + \frac{\omega_0^2}{K'^2}(1+\rho)\right]\frac{\partial D}{\partial X_2} + \frac{2i\omega_0(1+\rho)}{K'}\frac{\partial D}{\partial T_2}$$
$$-\frac{1+\rho}{K'}\frac{\partial^2 D}{\partial T_1^2} + \left[1 - \frac{(1+\rho)\,\omega_0^2}{K'^3}\right]\frac{\partial^2 D}{\partial X_1^2} - \frac{2\omega_0(1+\rho)}{K'^2}\frac{\partial^2 D}{\partial X_1\,\partial T_1} = -\bar{D}D^2\frac{8(1+\rho)}{K'}\Theta_E, \tag{19}$$

where

$$\Theta_E = -\frac{K'}{16(1+\rho)}\left[\Omega_E\left[4\omega_0^2(1-\rho) + \frac{4K'^2E_0^{(2)}E_0^{(1)}(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})}{(\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)})^2}(\tilde{\epsilon}^{(1)2} - 10\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)} + \tilde{\epsilon}^{(2)2})\right]\right.$$
$$\left. + 4\omega_0^2K'(1+\rho) - 3K'^4 + 32\frac{K'^3\tilde{\epsilon}^{(2)2}E_0^{(2)2}(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})}{(\tilde{\epsilon}^{(1)}+\tilde{\epsilon}^{(2)})^2}\right], \tag{20}$$

$$\Omega_E = \frac{K'(1-\rho)}{(1+\rho)(1-\rho-2K'^2)}\left[(1-\rho) - K'\alpha_E\left\{1 - \frac{(1+\rho)(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})}{(1-\rho)(\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)})}\right\} + K'^2\right]. \tag{21}$$

Equations (18) and (19) can be used to study the propagation of a finite-amplitude wavetrain over the surface. Using (15), (18) and (19) can be combined together to produce a nonlinear Schrödinger equation, by means of which one can study the stability of the system and obtain the electrohydrodynamic cutoff wavenumber.

3. The Schrödinger equations

As mentioned in §2, one can obtain a Schrödinger equation out of (18) and (19) with the help of (15) as follows. Differentiate equation (15) with respect to K' and substitute into (18); thus

$$\frac{\partial D}{\partial X_1} = -\frac{dK'}{d\omega_0}\frac{\partial D}{\partial T_1}, \tag{22}$$

and therefore

$$\frac{\partial^2 D}{\partial X_1^2} = \left(\frac{dK'}{d\omega_0}\right)^2\frac{\partial^2 D}{\partial T_1^2}, \quad \frac{\partial^2 D}{\partial X_1\,\partial T_1} = -\frac{dK'}{d\omega_0}\frac{\partial^2 D}{\partial T_1^2}. \tag{23}$$

Substituting in (19) with the help of (15) and replacing X_n and T_n by $\epsilon^n x$, $\epsilon^n t$ we get

$$\frac{\partial D}{\partial x} + \frac{dK'}{d\omega_0}\frac{\partial D}{\partial t} + \tfrac{1}{2}i\frac{d^2K'}{d\omega_0^2}\frac{\partial^2 D}{\partial t^2} = 8i\epsilon^2\hat{\theta}_E D^2\bar{D}, \tag{24}$$

where

$$\hat{\theta}_E = (1+\rho)\,\Theta_E\left[2K' - \alpha_E K' + (1+\rho)\frac{\omega_0^2}{K'}\right]^{-1}. \tag{25}$$

Equation (24) is valid for all wavenumbers larger than or equal to the cutoff wavenumber. It can be easily transformed into a Schrödinger equation by means of the transformation

$$\eta = x, \quad \tau = t - \frac{dK'}{d\omega_0}x,$$

and it then reads

$$\frac{\partial D}{\partial \eta}+\tfrac{1}{2}i\frac{d^2K'}{d\omega_0^2}\frac{\partial^2 D}{\partial \tau^2}=8i\epsilon^2\hat{\theta}_E D|D|^2. \tag{26}$$

Equation (26) is valid for all values of K' and hence can be used to obtain the cutoff wavenumber. However, from (20) and (21) we see that the Schrödinger equation is invalid when $\Omega_E \to \infty$. This is the resonance case and it occurs when $K' = K^*$, where

$$K^* = [\tfrac{1}{2}(1-\rho)]^{\frac{1}{2}}. \tag{27}$$

Equations (24) or (26) can also be used to obtain the surface elevation. Equation (24) admits the following solution for temporal variation

$$D = \tfrac{1}{2}b_0\, e^{is_0 t}+\text{const}, \tag{28}$$

where b_0 is a constant and

$$\tfrac{1}{2}s_0^2\frac{d^2K'}{d\omega_0^2}-s_0\frac{dK'}{d\omega_0}+2\epsilon^2\hat{\theta}_E\, b_0^2=0. \tag{29}$$

Hence

$$s_0=\frac{\dfrac{dK'}{d\omega_0}-\left[\left(\dfrac{dK'}{d\omega_0}\right)^2-4\epsilon^2\hat{\theta}_E\dfrac{d^2K'}{d\omega_0^2}b_0^2\right]^{\frac{1}{2}}}{\dfrac{d^2K'}{d\omega_0^2}}. \tag{30}$$

So (30) can be expanded, for ω_0 away from zero, since $dK'/d\omega_0 = O(1)$. Expanding and substituting for $dK'/d\omega_0$ and $\hat{\theta}_E$, we get

$$s_0=\epsilon^2 b_0^2\frac{\Theta_E}{\omega_0}. \tag{31}$$

Substituting for s_0 into D and hence for ξ, we finally obtain the following expression of the interface displacement ξ:

$$\xi(x,t)=\epsilon b_0\, cos\left[K'X_0-\left(\omega_0-\epsilon^2 b_0^2\frac{\Theta_E}{\omega_0}\right)t\right]$$
$$+\tfrac{1}{2}\epsilon^2 b_0^2\,\Omega_E\cos 2\left[K'X_0-\left(\omega_0-\epsilon^2 b_0^2\frac{\Theta_E}{\omega_0}\right)t\right]+O(\epsilon^3). \tag{32}$$

This solution is obtained from (24), which is a nonlinear equation having first- and second-order time derivatives. The space derivative involved in the equation is only a first-order one. As suggested by Davey (1972) and Nayfeh (1976) we can obtain an equation analogous to (24) which contains a second-order space derivative by inserting $d\omega_0/dK'$ instead of $dK'/d\omega_0$ into (19). With a similar procedure we get

$$\frac{\partial D}{\partial t}+\frac{d\omega_0}{dK'}\frac{\partial D}{\partial x}-\tfrac{1}{2}i\frac{d^2\omega_0}{dK'^2}\frac{\partial^2 D}{\partial x^2}=4i\epsilon^2\omega_0^{-1}\,\Theta_E\,\bar{D}D^2, \tag{33}$$

which is the analogous equation to (24). Equation (33) includes the first- and second-order spatial derivatives but involves a first-order time derivative only. One can show that (33) yields the same solution as given by (32) for the surface deflection.

Changing the independent variables from x and t into

$$\eta^*=x-\frac{d\omega_0}{dK'}t,\quad \tau^*=t,$$

(33) then becomes
$$\frac{\partial D}{\partial \tau^*}-\tfrac{1}{2}i\frac{d^2\omega_0}{dK'^2}\frac{\partial^2 D}{\partial \eta^{*2}} = 4i\epsilon^2\omega_0^{-1}\,\Theta_E\bar{D}D^2, \tag{34}$$
a second nonlinear Schrödinger equation. We observe that (34) is not valid when $\omega_0 = 0$ and therefore cannot be used to obtain the cutoff wavenumber. However, the advantage that (34) contains a second-order spatial derivative and first-order time derivative will help considerably in establishing a stability criterion. In the absence of the electric field, (26) and (34) are the same as those obtained by Nayfeh (1976).

4. The cutoff wavenumber

As mentioned in §3, we can use (26) to obtain the cutoff wavenumber. In the limit as $\omega_0 \to 0$ we find
$$\frac{dK'}{d\omega_0} \to 2\omega_0(1+\rho)\,[K'(2K'-\alpha_E)]^{-1}, \tag{35}$$
$$\hat{\theta}_E \to -\frac{K'^3}{16(2K'-\alpha_E)}\left[\frac{4K'(\tilde{\epsilon}^{(1)2}-10\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(2)2})}{(\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)})^2(1-\rho-2K'^2)}\alpha_E^2-3K' + 32\frac{\tilde{\epsilon}^{(2)}\tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)2}-\tilde{\epsilon}^{(1)2}}\alpha_E\right]. \tag{36}$$
Hence (30) tends to
$$s_0 = \omega_0-\left[\omega_0^2-2\epsilon^2b_0^2\frac{K'(2K'-\alpha_E)}{1+\rho}\hat{\theta}_E\right]^{\frac{1}{2}}. \tag{37}$$
Since ω_0 is near zero, s_0 cannot be expanded as given by (31), and ξ should be substituted from (37) to yield
$$\xi(x,t) = \epsilon b_0\cos\left[K'x_0-\left(\omega_0^2-2\epsilon^2b_0^2\frac{K'(2K'-\alpha_E)}{1+\rho}\hat{\theta}_E\right)^{\frac{1}{2}}t\right] + \tfrac{1}{2}\epsilon^2b_0^2\,\Omega_E\cos 2\left[K'x_0-\left(\omega_0^2-2\epsilon^2b_0^2\frac{K'(2K'-\alpha_E)}{1+\rho}\hat{\theta}_E\right)^{\frac{1}{2}}t\right]+O(\epsilon^3). \tag{38}$$
Equation (37) shows that the cutoff wavenumber may be given by the relation
$$\omega_0^2 = 2\epsilon^2b_0^2\frac{K'(2K'-\alpha_E)}{1+\rho}\hat{\theta}_E. \tag{39}$$
Recalling the value of $\hat{\theta}_E$ from (36) and using (15), we get for (39)
$$6\epsilon^2b_0^2K'^6-2\epsilon^2b_0^2d_2K'^5-[16+\epsilon^2b_0^2(-4d_1+3(1-\rho))]K'^4 + [16\alpha_E+\epsilon^2b_0^2d_2(1-\rho)]K'^3-8(1-\rho)K'^2-8\alpha_E(1-\rho)K'+8(1-\rho)^2 = 0, \tag{40}$$
where
$$d_1 = \frac{\tilde{\epsilon}^{(1)2}-10\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(2)2}}{(\tilde{\epsilon}^{(1)}+\tilde{\epsilon}^{(2)})^2}4(\rho-1)\sinh^2\tilde{\theta}_E, \tag{41}$$
$$d_2 = 32\frac{\tilde{\epsilon}^{(2)}\tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)2}-\tilde{\epsilon}^{(1)2}}2(\rho-1)^{\frac{1}{2}}\sinh\tilde{\theta}_E. \tag{42}$$
The solution of the resulting equation for K' (i.e. (40)) leads to the cutoff wavenumber, which depends on the electric field and amplitude. If we put
$$K' = K_0+\epsilon^2K_2+O(\epsilon^3), \tag{43}$$

from (43) and (40)–(42) we get

$$K_0 = K'_{cE} = (\rho-1)^{\frac{1}{2}}[\sinh\tilde{\theta}_E + \cosh\tilde{\theta}_E], \tag{44}$$

which is in agreement with the linear case. Also we get

$$K_2 = \tfrac{3}{16}b_0^2 K_0^3 \frac{2K_0^3 - \frac{2}{3}d_2 K_0^2 + ((\rho-1)+\frac{4}{3}d_1) K_0 - \frac{1}{3}d_2(\rho-1)}{4K_0^3 - 2K_0^2\alpha_E - (\rho-1) K_0 - \frac{1}{2}\alpha_E(\rho-1)}. \tag{45}$$

The linear cutoff wavenumber receives a second-order increment $\epsilon^2 K_2$. The nonlinearity effect is stabilizing if K_2 is negative and *vice versa*. The sign of K_2, as given by (45), depends on the equilibrium electric field and the values of the dielectric constants of the two fluids, since they decide the sign of the coefficients d_1 and d_2. Thus we can explain Taylor's observation of stable oscillations near the linear cutoff wavenumber.

5. Stability analysis

The analysis of this section will be based on (33). It was shown by Strauss (1979) that the solution of the Schrödinger equation (34) is bounded provided that

$$\Theta_E \frac{d^2\omega_0}{dK'^2} \leqslant 0. \tag{46}$$

Thus a finite-amplitude wave propagating through the surface is stable when the condition given by (46) is satisfied.

Many investigators, such as Karpman & Krushkal (1969), Hasimoto & Ono (1972), Nayfeh (1976) and Strauss (1979), have examined the linear stability of a finite-amplitude wavetrain propagating through the interface by linearly perturbing the solution of (33). They found that the linear stability implies the same condition (46). Combining these two results one can deduce an important property of the finite-amplitude wave-train. It is evident that, if such a finite-amplitude wave exists, then it is linearly stable.

Before we examine condition (46) in general we shall discuss some limiting cases.

First we observe that the quantities Θ_E and $d^2\omega_0/dK'^2$ have the same sign in the limit as both T and K' approach zero for $\rho < 1$. Therefore electrohydrodynamic Stokes waves are unstable for $\rho < 1$. The electric field has no effect in this case.

For capillary waves both T ansd $K' \to \infty$. The system is stable if

$$\rho > 3 - 2\sqrt{2}. \tag{47}$$

The above condition is independent of the electric field. Therefore the electric field has no effect for the present case too.

In general, the sign of Θ_E and $d^2\omega_0/dK'^2$ depends on $E^{(1)}$, $E^{(2)}$, $\tilde{\epsilon}^{(1)}$ and $\tilde{\epsilon}^{(2)}$. A stability chart can be drawn whose boundary curves are given by $\Theta_E = 0$ and $d^2\omega_0/dK'^2 = 0$. The two equations, on substitution for Θ_E and ω_0, take the form

$$3K'^4 - 8(\rho-1)^{\frac{1}{2}} K'^3 \sinh\tilde{\theta}_E + 6K'^2(1-\rho) - (1-\rho)^2 = 0, \tag{48}$$

$$\begin{aligned} &2K'^4[2(1-\rho)^2 - (1+\rho)^2] + 4K'^3[h_1\alpha_E(1+\rho)(1-\rho) - \alpha_E(1-\rho)^2 \\ &- h_3(1-\rho)\alpha_E - 2(1+\rho)^2\alpha_E(8h_2-1)] + K'^2[8(1-\rho)^3 \\ &- 4\alpha_E^2 h_3 h_1(1+\rho) + 4\alpha_E^2 h_3(1-\rho) - 7(1-\rho)(1+\rho)^2] \\ &+ 4K'(1-\rho)[-h_3\alpha_E(1-\rho) + h_1\alpha_E(1+\rho)(1-\rho) \\ &- \alpha_E(1-\rho)^2 + \alpha_E(8h_2-1)(1+\rho)^2] + 4(1-\rho)^2[(1-\rho)^2 + (1+\rho)^2] = 0, \end{aligned} \tag{49}$$

where
$$h_1 = \frac{\tilde{\epsilon}^{(1)2}+\tilde{\epsilon}^{(2)2}-10\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)}}{\tilde{\epsilon}^{(2)2}-\tilde{\epsilon}^{(1)2}}, \quad h_2 = \frac{\tilde{\epsilon}^{(2)}\tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)2}-\tilde{\epsilon}^{(1)2}}, \tag{50}$$

$$h_3 = (1-\rho)-(1+\rho)\frac{\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)}}. \tag{51}$$

We may also observe that the curve
$$K'^2 = \tfrac{1}{2}(1-\rho) \tag{52}$$
is a third transition curve, because Ω_E, and accordingly Θ_E, changes sign across this curve. The third curve represents the second harmonic resonance. Since the relation is independent of the electric field, the curve will be the same for all the cases discussed below.

We see from (52), (53) that the signs of the factors h_1, h_2, h_3 depend on the sign of $\tilde{\epsilon}^{(1)}-\tilde{\epsilon}^{(2)}$. Therefore the number of real positive roots of (49) is governed by the properties of the dielectric constant of the upper and lower fluids.

Accordingly the branches of the curves $\Theta_E = 0$, and consequently the stability criterion, will strongly depend on whether the lower fluid or the upper fluid has a greater dielectric constant. The result is in contrast with the linear theory, where the sign of $\tilde{\epsilon}^{(1)}-\tilde{\epsilon}^{(2)}$ has no implications for the stability criterion.

A similar phenomenon was observed in the surface instability of ferromagnetic fluids in a vertical magnetic field. Owing to nonlinear effects, Galitis (1969) found that the instability depends on the magnetic permeability. He used the boundary condition $\mu^{(1)}H_n^{(1)} = \mu^{(2)}H_n^{(2)}$, which is analogous to our condition $\tilde{\epsilon}^{(1)}E_n^{(1)} = \tilde{\epsilon}^{(2)}E_n^{(2)}$ given by (2). The above result is mathematically expected. The electrohydrodynamic body force is
$$-\nabla\Pi+q\mathbf{E}-\tfrac{1}{2}E^2\nabla\epsilon.$$

Since both of the fluids are homogeneous, the electric body force vanishes in the bulk of the fluids. The condition of continuity of the normal stress is simply obtained by integrating the body force across the interface. The jump in the values of $\tilde{\epsilon}$ and E_0 will then contribute to the normal stress. The effect did not appear in the linear theory because the body force is nonlinear.

The stability analysis may be understood by studying the stability graphs represented by (48)–(52).

The curves in figures 2–4 correspond to the case $\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)}$ while the curves in figures 5–7 correspond to the case $\tilde{\epsilon}^{(1)} < \tilde{\epsilon}^{(2)}$.

In the limit as $\alpha_E \to 0$ i.e. in absence of electric field) the curves reduce to those corresponding to the pure hydrodynamical case obtained by Nayfeh (1976). The curves for this case ($\alpha_E \to 0$) are shown in figure 1. They are also drawn in broken lines in figure 2. The curves to the right of any of these graphs represent the linear case. As $\alpha_E \to 0$, Nayfeh showed that as the amplitude increases the curves shift to the left and two regions of subharmonic resonance appear. For $\rho = 0$, stability is only possible for values of K' between 0·393 and 0·707, approximately. As α_E increases, the stability regions are redistributed due to the presence of the electric field. The changes of loci of $\Theta_E = 0$, $\omega_0'' = 0$ produce new stable and unstable regions.

We plot curves for K' against ρ for constant values of α_E and also K' versus α_E for constant values of ρ. For all values of α_E, the transition curve $K'^2 = \frac{1}{2}(1-\rho)$ is the same and coincides with the corresponding curve for $\alpha_E \to 0$. The curve $\omega_0'' = 0$ shifts above as α_E increases, producing more unstable areas. The behaviour of the curve $\Theta_E = 0$ depends on the value of α_E and the sign of $\tilde{\epsilon}^{(1)}-\tilde{\epsilon}^{(2)}$.

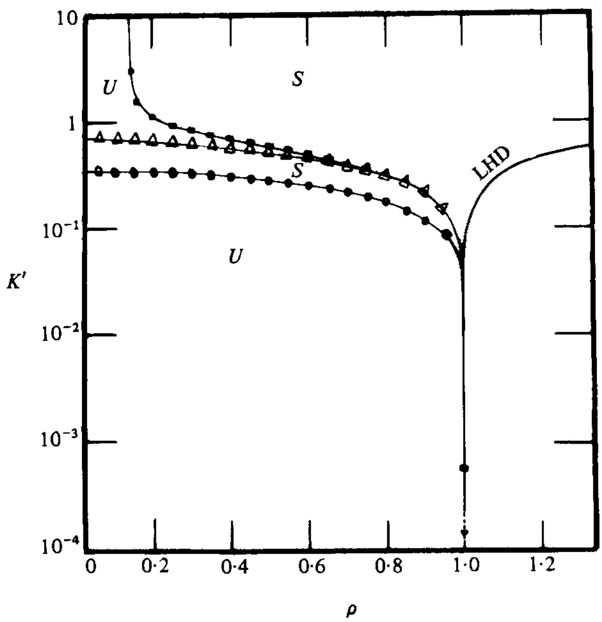

FIGURE 1. Stability diagram for $\alpha_E = 0$ (hydrodynamical case). Symbol ● represents the curve $\Theta_E|_{\alpha_E=0} = 0$. The symbol ◒ represents the curve $(d^2\omega_0/dK'^2)|_{\alpha_E=0} = 0$, the symbol △ represents the curve $K'^2 = \frac{1}{2}(1-\rho)$, LHD refers to linear hydrodynamic curve $K' = (\rho-1)^{\frac{1}{2}}$, U refers to unstable region and S refers to stable region.

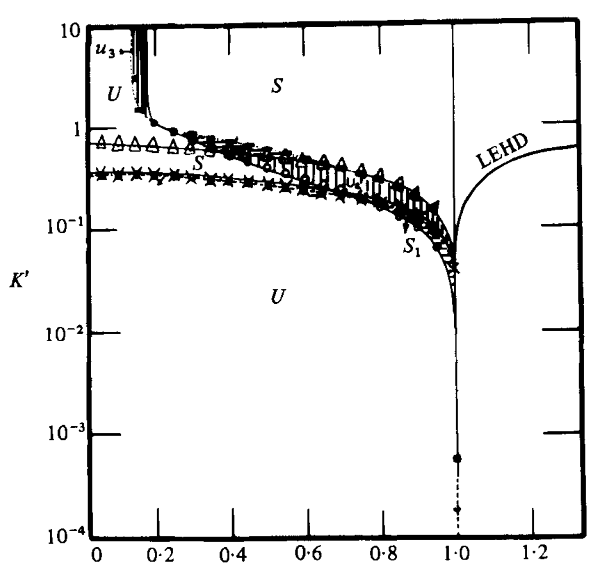

FIGURE 2. Stability diagram for $\alpha_E = 0{\cdot}031\,936\,1$ with $\bar{\epsilon}^{(1)} > \bar{\epsilon}^{(2)}$. The symbol ○ represents the curve $\Theta_E = 0$, the symbol × represents the curve $d^2\omega_0/dK'^2 = 0$, LEHD refers to the curve $\omega_0 = 0$; the shaded regions are U_1, U_2, S_1, U_3 and S_2, and represent the newly formed regions due to the presence of electric field.

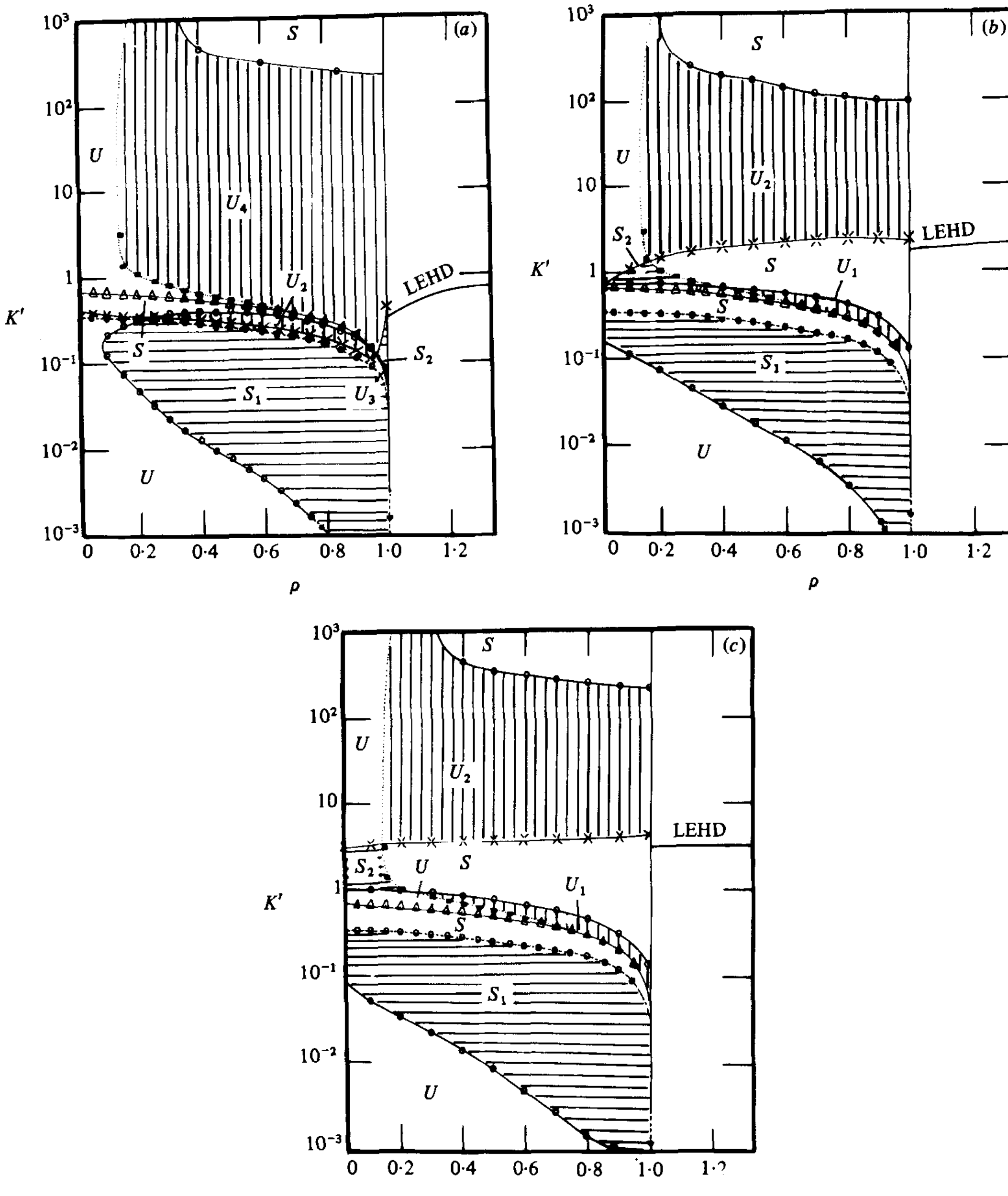

FIGURE 3. Stability diagram in the (K', ρ)-plane. The (a) refers to the case $\alpha_E = 0{\cdot}344083$ $(\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)})$, the letter (b) refers to the case $\alpha_E = 1{\cdot}94554$ $(\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)})$, while (c) corresponds to $\alpha_E = 3{\cdot}04913$ $(\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)})$. The dotted curves refer to $\alpha_E = 0$. The symbol $\odot$ refers to the loci of $\Theta_E = 0$, the symbol $\triangle$ refers to the curve $K'^2 = \frac{1}{2}(1-\rho)$, the symbol $\times$ refers to the curve $d^2\omega_0/dK'^2 = 0$.

Figures 2, 3 represent the variations of the stability charts in the (K', ρ)-plane, for various values of α_E, for the case $\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)}$. Comparing these graphs with the case $\alpha_E \rightarrow 0$, we see that for small values of α_E (figure 2, where $\alpha_E = 0{\cdot}03193$), the $\Theta_E = 0$ curve diverges slightly to the left ($K' < 1$) forming stable regions for $K' > [\frac{1}{2}[(1-\rho)]^{\frac{1}{2}}$, $K' > K'(\Theta_E | \alpha_{E=0} = 0)$. We also observe that the first-subharmonic-resonance region is nearly the same as in the absence of electric field except for $\rho > 0{\cdot}75$ where the

α_E	Lower limit of K'	Upper limit of K'
0	0·393	0·707
0·031	0·394	0·707
0·344	0·410	0·707
1·9455	0·179, 0·708	0·707, 0·753
3·049	0·0949, 1·095	0·707, 3·5022

TABLE 1. Range of stable values of K' at $\rho = 0$, $\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)}$

region decreases. The second-harmonic-resonance region is now composed of two joined portions. The right portion $K' < [\frac{1}{2}(1-\rho)]^{\frac{1}{2}}$ was stable when $\alpha_E = 0$. For $\rho = 0$ the system is stable only for $0{\cdot}394 < K' < 0{\cdot}707$.

As α_E increases, the curve $\Theta_E = 0$ forms a loop which expands gradually to suppress a bigger part of the first unstable region, and the field is then stabilizing. This influence increases as ρ increases. However, for higher values of K' the field has a destabilizing effect, as shown in figures 2, 3. The shaded regions are the regions which are changed from stable to unstable regions or *vice versa* owing to the presence of the electric field.

It is interesting to examine the range of K' at $\rho = 0$ which yields stability for various values of α_E. For the last two values of α_E in table 1, two regions of stability are possible. The reason is that the loop formed by $\Theta_E = 0$ is enlarged as α_E increases until it crosses the K' axis, producing two regions of stability. When the curve $\omega_0'' = 0$ lies above the curve $K'^2 = \frac{1}{2}(1-\rho)$, a portion of the second harmonic resonance corresponding to $K' < K'(\Theta_E)$ is stabilized. Thus the electric field plays a dual role when $\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)}$, suppressing subharmonic resonance for smaller values of K', and destabilizing for larger values of K'.

The above results are more obvious if we study the stability charts in the (K', α_E)-plane (figure 4) that represents the case $\rho = 0$. For values of $\alpha_{E1} < 1{\cdot}945$, stability is possible provided that the values of K' lie within the region S_1. The range of K' decreases as α_{E1} increases, keeping $K' = 0{\cdot}707$ as an upper bound. For $\alpha_{E1} > 1{\cdot}945$ two regions of stability appear: (S, S'). The lower region has $K' = 0{\cdot}707$ as an upper bound. S' increases as α_E increases starting from the positive root $(\alpha_{E1}, \alpha_{E2})$ of the equation

$$2h_3\,\alpha_{E2}^2(1-h_1) + 3\sqrt{2}\,\alpha_{E2}(h_1 - h_3 - 1) + 9 = 0, \tag{53}$$

where h_1, h_2 and h_3 are given by (50), (51), and $\rho = 0$. It is clear that the field is stabilizing for $\alpha_{E1} < \alpha_E < \alpha_{E2}$. For $\alpha_E > \alpha_{E2}$ the field is still stabilizing for

$$K'(\Theta_E|_{\rho=0}) < K' < \sqrt{\tfrac{1}{2}}.$$

On the other hand, when $\tilde{\epsilon}^{(1)} < \tilde{\epsilon}^{(2)}$ (figures 5, 6), the curve $\omega_0'' = 0$ behaves in a manner similar to the previous case. The curve $\Theta_E = 0$ changes its behaviour as α_E increases, but does not form a loop in the first quadrant for $\rho < 1$. Therefore the first unstable region cannot be stabilized significantly for the present case, except for a small band near $\rho = 0$. The first stable region is disturbed and an unstable portion appears for a smaller value of ρ. When the curve $\omega_0'' = 0$ lies below the curve $K'^2 = \frac{1}{2}(1-\rho)$, a stable region appears above the latter curve instead of the infinite strip of the second unstable region, and the field is then stabilizing for higher values of K' $(K' > [\frac{1}{2}(1-\rho)]^{\frac{1}{2}})$. When α_E increases, the curve $\omega_0'' = 0$ shifts above the curve

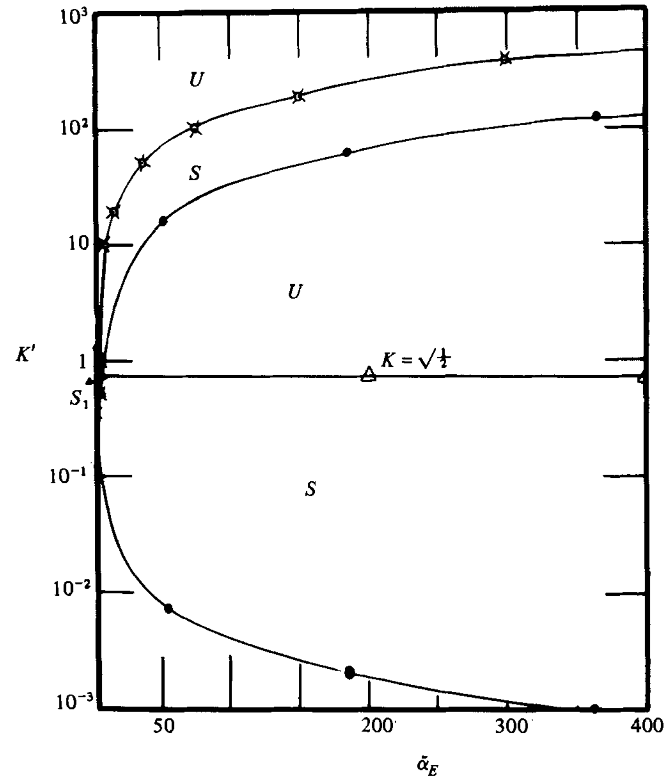

FIGURE 4. Stability diagram in the (K', α_E)-plane for $\rho = 0$ with $\tilde{\epsilon}^{(1)} > \tilde{\epsilon}^{(2)}$. The symbol ⌧ represents the curve $d^2\omega_0/dK'^2 = 0$, the symbol ● represents the curve $\Theta_E = 0$, and the symbol △ represents the curve $K' = \sqrt{\frac{1}{2}}$.

	First region		Second region	
α_E	Lower limit of K'	Upper limit	Lower limit	Upper limit
0	0·393	0·707	—	—
0·0319	0·394	0·707	—	—
0·0877	0·370	0·385	—	—
0·1452	0·280	0·390	—	—
0·203	0·210	0·420	—	—
0·2605	0·205	0·410	—	—
0·320	0·408	0·507	0·707	35·189
0·908	0·434	0·448	0·707	99·842
1·795	0·562	0·601	0·707	197·442
2·982	0·707	0·904	3·396	327·994
4·469	0·707	1·375	5·604	491·499

TABLE 2. Ranges of stable regions of K' for $\rho = 0$, $\tilde{\epsilon}^{(1)} < \tilde{\epsilon}^{(2)}$

$K' = [\frac{1}{2}(1-\rho)]^{\frac{1}{2}}$, producing an unstable region, though it stabilizes the system for values of K' ($K' > K'(\omega''_{\alpha_E=0})$, $\rho < \rho(\Theta_{\alpha_E=0})$). Therefore the electric field is stabilizing for larger values of K' for a range of smaller values of ρ, and destabilizing for smaller values of K'. In table 2 we calculate the range for K' yielding stability for $\rho = 0$ for various values of α_E, when $\tilde{\epsilon}^{(1)} < \tilde{\epsilon}^{(2)}$.

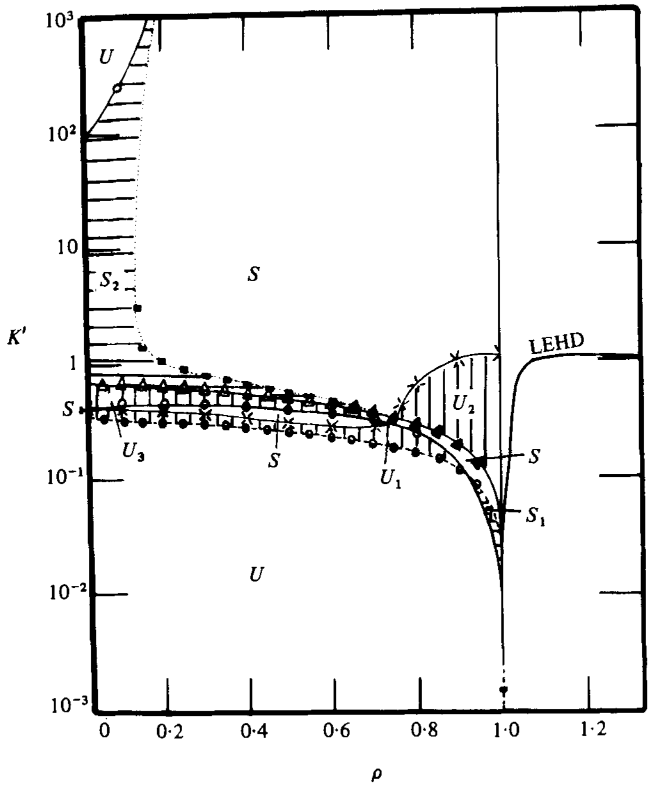

FIGURE 5. The case $\alpha_E = 0{\cdot}907\,935$ with $\tilde{\epsilon}^{(2)} > \tilde{\epsilon}^{(1)}$. The symbols ○ represent the curve $\Theta_E = 0$, × represents $d^2\omega_0/dK'^2 = 0$, and △ represents $K'^2 = \frac{1}{2}(1-\rho)$. Shaded regions are gain regions due to the presence of the electric field.

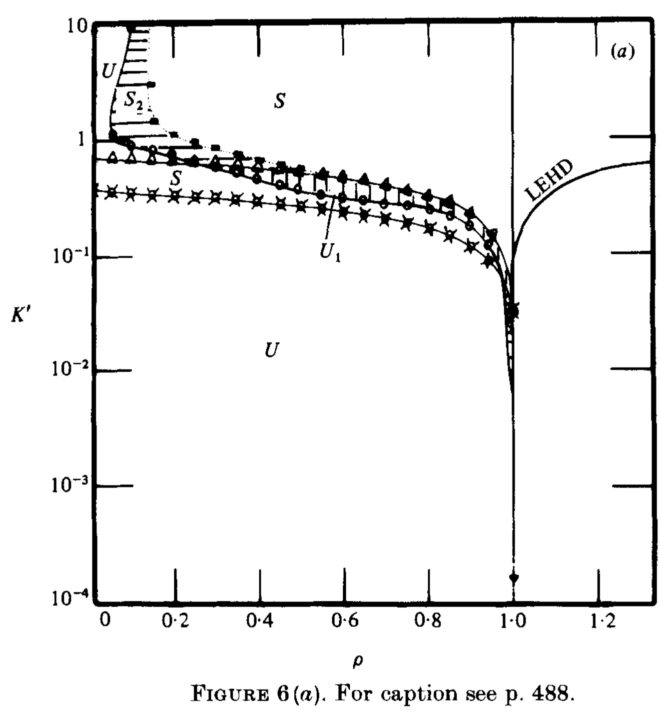

FIGURE 6(*a*). For caption see p. 488.

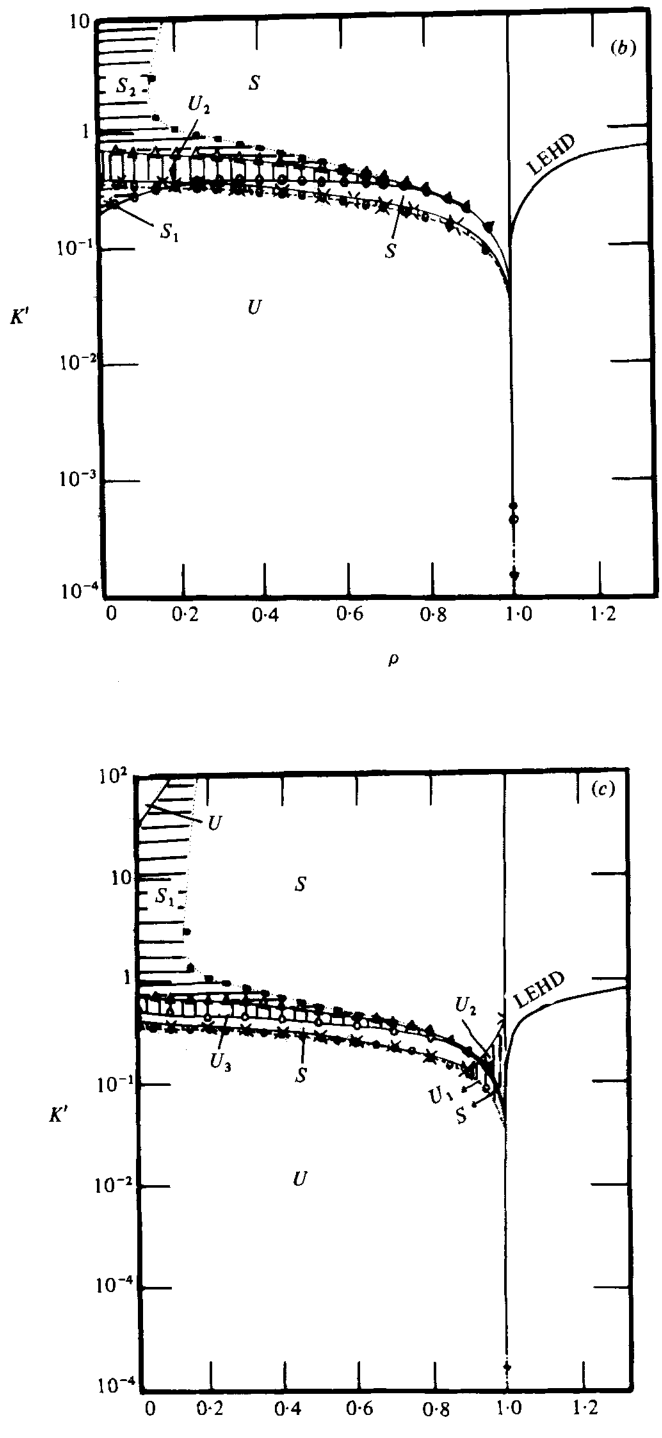

FIGURE 6(*b*, *c*). For caption see p. 488.

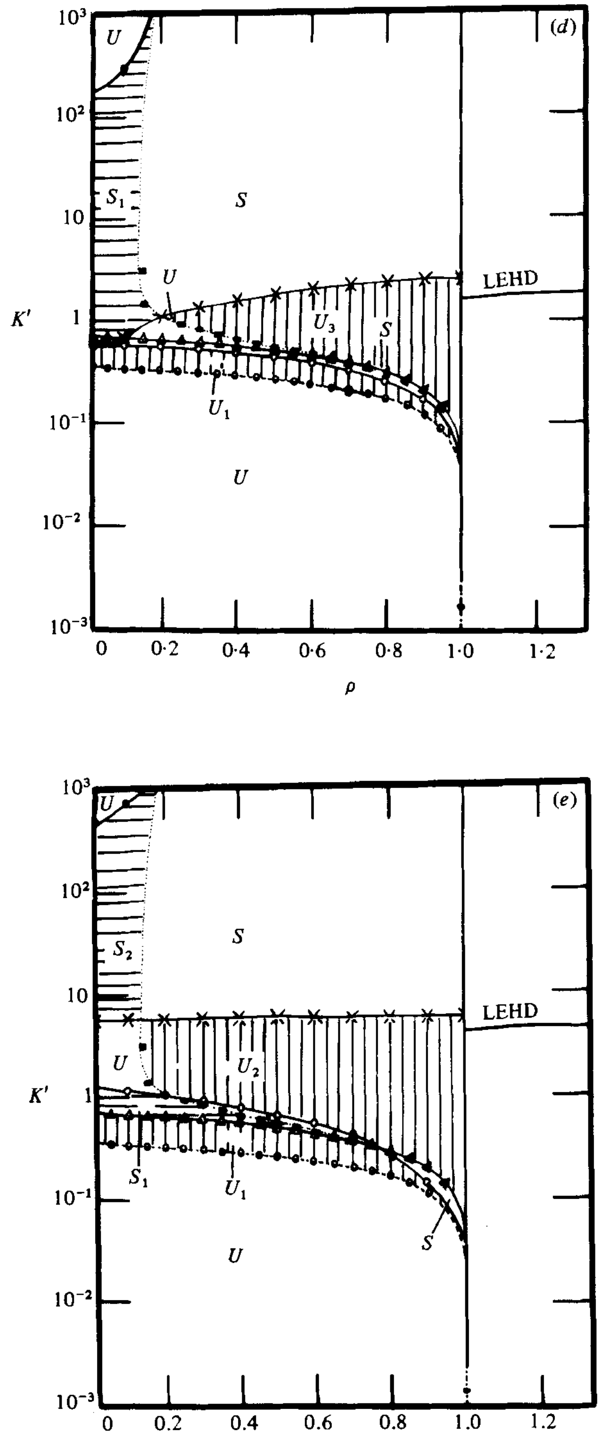

FIGURE 6. The (K', ρ)-plane for $\tilde{\epsilon}^{(2)} > \tilde{\epsilon}^{(1)}$. The (a) refers to case $\alpha_E = 0{\cdot}031\,936$; (b) to $\alpha_E = 0{\cdot}202\,885$; (c) to $\alpha_E = 0{\cdot}320\,108$; (d) to $\alpha_E = 1{\cdot}795\,41$ and (e) to $\alpha_E = 4{\cdot}469\,34$. The symbols ○, ×, △ are as in figure 5.

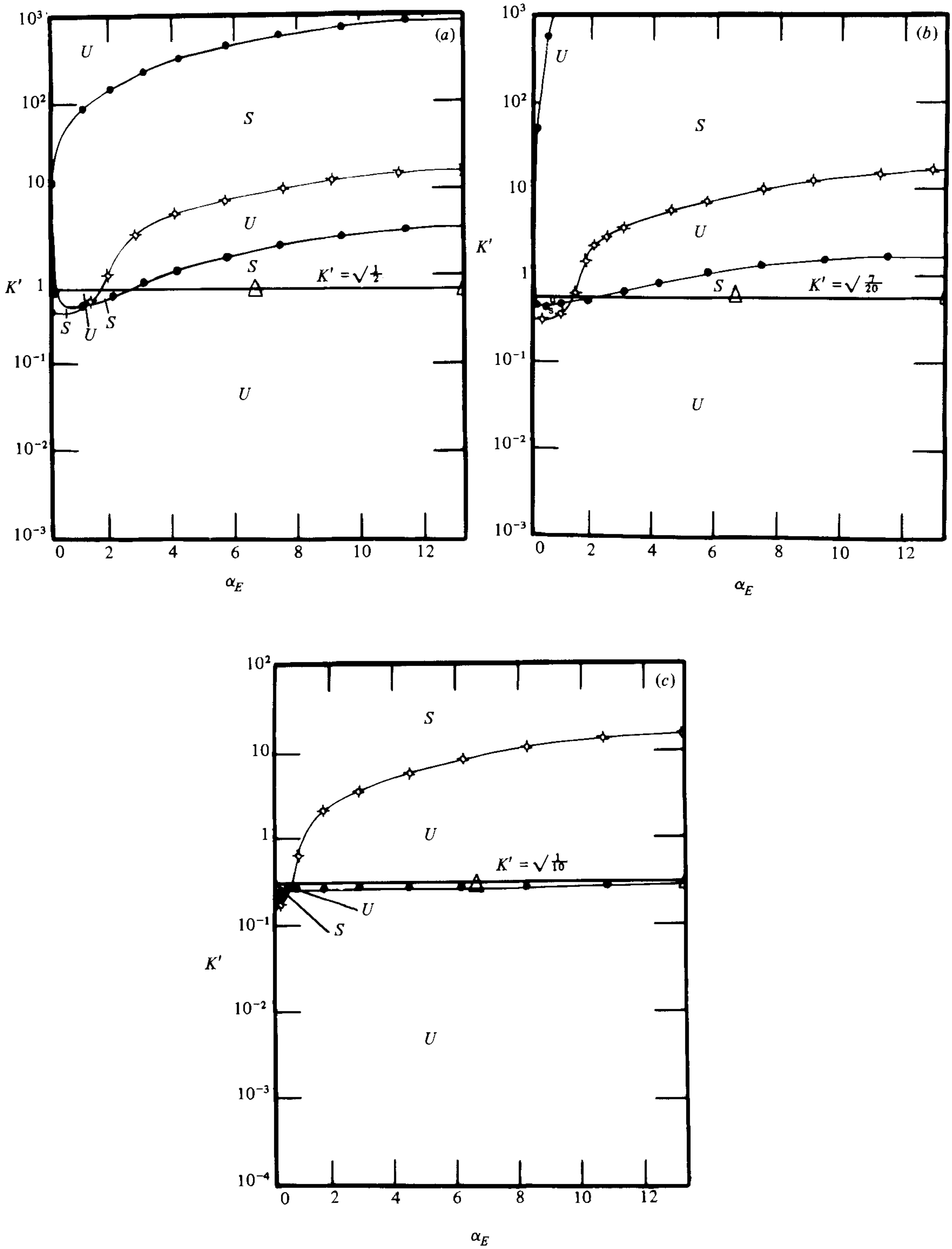

FIGURE 7. The (K, α_E)-plane for $\bar{\epsilon}^{(2)} > \bar{\epsilon}^{(1)}$. The (*a*) refers to $\rho = 0$; (*b*) to $\rho = 0{\cdot}3$ and (*c*) to $\rho = 0{\cdot}8$. The symbols -◇-, ● are as in figure 4, and △ represents the curve $K'^2 = \frac{1}{2}(1-\rho)$.

The discussion becomes more obvious when we examine the (K, α_E)-plane (figure 7) for $\tilde{\epsilon}^{(1)} < \tilde{\epsilon}^{(2)}$. Figure 7 is the stability diagram corresponding to the cases $\rho = 0$, $\rho = 0{\cdot}3$, $\rho = 0{\cdot}8$. In absence of electric field, $(\alpha_E = 0)$ stability is only possible when $0{\cdot}393 < K' < 0{\cdot}707$. As α_E increases, two stable regions appear because the curve $\Theta_E(\alpha_E) = 0$ has two branches. The stabilizing influence increases considerably as α_E exceeds approximately 2·5.

For $\rho = 0{\cdot}3$ we see that the curve $\Theta_E = 0$, with the two transition curves $\omega_0'' = 0$ and $K' = \sqrt{\frac{7}{20}}$, produces two stable regions below the curve $K' = \sqrt{\frac{7}{20}}$ and two stable regions above it. For the first two stable regions, the larger region gradually dies down when $\alpha_E \simeq 1{\cdot}5$ and the smaller region also dies down when $\alpha_E \simeq 2{\cdot}5$. We see that the field is relatively destabilizing for $K' < \sqrt{\frac{7}{20}}$. But for the second two stable regions, i.e. for $K' > \sqrt{\frac{7}{20}}$, the regions grow with increasing α_E. For $\rho = 0{\cdot}8$ we see that Θ_E lies entirely below the curve $K' = \sqrt{\frac{1}{10}}$ for $\alpha_E > 0$, and makes two stable regions; one of the regions lies between $K' > \omega_0''$ and $K' < \Theta_E$ for small α_E, i.e. $0 < \alpha_E < 0{\cdot}5$, and the other region lies between $K' > \Theta_E$ and $K' < \sqrt{\frac{1}{10}}$ for $\alpha_E \geqslant 0{\cdot}5$. Also the curve of ω_0'' forms a stable region for $K' > \omega_0''$ and $K' > \sqrt{\frac{1}{10}}$ for all values of α_E. In this figure we see that the field destabilizes for the two regions $\sqrt{\frac{1}{10}} > K' > \Theta_E$ for small α_E and $K' > \sqrt{\frac{1}{10}}$, $K' < \omega_0''$ when $\alpha_E \geqslant 0{\cdot}5$. The instability regions have larger ranges for larger values of α_E.

From the numerical discussion, it is evident that, apart from the effect of the variation of the dielectric constant, the electric field plays a dual role in the stability criterion in contrast with the linear theory. It seems in general that electrohydrodynamics is a nonlinear phenomenon and it is better understood via nonlinear analysis.

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Appendix

If we substitute for ξ, ψ, ϕ, as given by (11)–(14), into (1)–(10) and equate terms of equal powers of ϵ, we get, for terms of order ϵ, the set of equations

$$\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \psi_1^{(2),(1)}}{\partial y^2} = 0, \tag{A 1}$$

$$\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \phi_1^{(2),(1)}}{\partial y^2} = 0, \tag{A 2}$$

$$\left\{\!\!\left|\frac{\partial \psi_1}{\partial X_0}\right|\!\!\right\} = \frac{\partial \xi_1}{\partial X_0}\{\!|E_0|\!\} \quad \text{at} \quad y = 0, \tag{A 3}$$

$$\left\{\!\!\left|\tilde{\epsilon}\frac{\partial \psi_1}{\partial y}\right|\!\!\right\} = 0 \quad \text{at} \quad y = 0, \tag{A 4}$$

$$\frac{\partial \xi_1}{\partial T_0} - \frac{\partial \phi_1^{(2),(1)}}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{A 5}$$

$$\frac{\partial \phi_1^{(1)}}{\partial T_0} - \rho\frac{\partial \phi_1^{(2)}}{\partial T_0} + (1-\rho)\,\xi_1 - \frac{\partial^2 \xi_1}{\partial X_0^2} = -\left\{\!\!\left|\tilde{\epsilon}E_0\frac{\partial \psi_1}{\partial y}\right|\!\!\right\} \quad \text{at} \quad y = 0, \tag{A 6}$$

with the following solutions:

$$\xi_1 = D(X_1, X_2, T_1, T_2)\, e^{i(K'X_0-\omega_0 T_0)} + \bar{D}(X_1, X_2, T_1, T_2)\, e^{-i(K'X_0-\omega_0 T_0)}. \tag{A 7}$$

$$\psi_1^{(2)} = -\frac{E_0^{(2)}(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})}{\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)}}\left[D\, e^{i(KX_0-\omega_0 T_0)-K'y} + \bar{D}\, e^{-i(K'X_0-\omega_0 T_0)-K'y}\right], \tag{A 8}$$

$$\psi_1^{(1)} = \frac{E_0^{(1)}(\tilde{\epsilon}^{(2)}-\tilde{\epsilon}^{(1)})}{\tilde{\epsilon}^{(2)}+\tilde{\epsilon}^{(1)}}\left[D\, e^{i(K'X_0-\omega_0 T_0)+K'y} + \bar{D}\, e^{-i(K'X_0-\omega_0 T_0)+K'y}\right], \tag{A 9}$$

$$\phi_1^{(2)} = \frac{i\omega_0}{K'}\left[D\, e^{i(K'X_0-\omega_0 T_0)-K'y} - \bar{D}\, e^{-i(K'X_0-\omega_0 T_0)-K'y}\right], \tag{A 10}$$

$$\phi_1^{(1)} = \frac{-i\omega_0}{K'}\left[D\, e^{i(K'X_0-\omega_0 T_0)+K'y} - \bar{D}\, e^{-i(K'X_0-\omega_0 T_0)+K'y}\right], \tag{A 11}$$

where $\bar{D}(X_1, X_2, T_1, T_2)$ is the complex conjugate of $D(X_1, X_2, T_1, T_2)$. The above solutions should be bounded by the condition given by (15) (the linear dispersion equation).

At $O(\epsilon^2)$ one finds the set

$$\frac{\partial^2 \psi_2^{(2),(1)}}{\partial X_0^2} + \frac{\partial \psi_2^{(2),(1)}}{\partial y^2} = -2\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_0\, \partial X_1}, \tag{A 12}$$

$$\frac{\partial^2 \phi_2^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \phi_2^{(2),(1)}}{\partial y^2} = -2\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0\, \partial X_1}, \tag{A 13}$$

$$\begin{aligned}\left\{\!\!\left\{\frac{\partial \psi_2}{\partial X_0}\right\}\!\!\right\} &= \left(\frac{\partial \xi_2}{\partial X_0} + \frac{\partial \xi_1}{\partial X_1}\right)\{\!\{E_0\}\!\} \\ &\quad - \left\{\!\!\left\{\frac{\partial \psi_1}{\partial X_1}\right\}\!\!\right\} - \xi_1\left\{\!\!\left\{\frac{\partial^2 \psi_1}{\partial y\, \partial X_0}\right\}\!\!\right\} - \frac{\partial \xi_1}{\partial X_0}\left\{\!\!\left\{\frac{\partial \psi_1}{\partial y}\right\}\!\!\right\} \quad \text{at} \quad y = 0,\end{aligned} \tag{A 14}$$

$$\left\{\!\!\left\{\tilde{\epsilon}\frac{\partial \psi_2}{\partial y}\right\}\!\!\right\} = \frac{\partial \xi_1}{\partial X_0}\left\{\!\!\left\{\tilde{\epsilon}\frac{\partial^2 \psi_1}{\partial X_0}\right\}\!\!\right\} - \xi_1\left\{\!\!\left\{\tilde{\epsilon}\frac{\partial^2 \psi_1}{\partial y^2}\right\}\!\!\right\} \quad \text{at} \quad y = 0, \tag{A 15}$$

$$\frac{\partial \xi_2}{\partial T_0} - \frac{\partial \phi_2^{(2),(1)}}{\partial y} = -\frac{\partial \xi_1}{\partial T_1} + \xi_1\frac{\partial^2 \phi_1^{(2),(1)}}{\partial y^2} - \frac{\partial \phi_1^{(2),(1)}}{\partial X_0}\frac{\partial \xi_1}{\partial X_0} \quad \text{at} \quad y = 0, \tag{A 16}$$

$$\begin{aligned}&\frac{\partial \phi_2^{(1)}}{\partial T_0} - \rho\frac{\partial \phi_2^{(2)}}{\partial T_0} + (1-\rho)\,\xi_2 - \frac{\partial^2 \xi_2}{\partial X_0^2} + \left\{\!\!\left\{\tilde{\epsilon}E_0\frac{\partial \psi_2}{\partial y}\right\}\!\!\right\} \\ &\quad = -\xi_1\frac{\partial^2 \phi_1^{(1)}}{\partial T_0\, \partial y} - \frac{\partial \phi_1^{(1)}}{\partial T_1} + \rho\xi_1\frac{\partial^2 \phi_1^{(2)}}{\partial T_0\, \partial y} + \rho\frac{\partial \phi_1^{(2)}}{\partial T_1} \\ &\qquad - \frac{1}{2}\left(\frac{\partial \phi_1^{(1)}}{\partial X_0}\right)^2 + \tfrac{1}{2}\rho\left(\frac{\partial \phi_1^{(2)}}{\partial X_0}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi_1^{(1)}}{\partial y}\right)^2 \\ &\qquad + \tfrac{1}{2}\rho\left(\frac{\partial \phi_1^{(2)}}{\partial y}\right)^2 + 2\frac{\partial^2 \xi_1}{\partial X_0\, \partial X_1} - \frac{1}{2}\left\{\!\!\left\{\tilde{\epsilon}\left(\frac{\partial \psi_1}{\partial X_0}\right)^2\right\}\!\!\right\} \\ &\qquad - \xi_1\left\{\!\!\left\{\tilde{\epsilon}E_0\frac{\partial^2 \psi_1}{\partial y^2}\right\}\!\!\right\} + \frac{1}{2}\left\{\!\!\left\{\tilde{\epsilon}\left(\frac{\partial \psi_1}{\partial y}\right)^2\right\}\!\!\right\} + 2\frac{\partial \xi_1}{\partial X_0}\left\{\!\!\left\{\tilde{\epsilon}E_0\frac{\partial \psi_1}{\partial X_0}\right\}\!\!\right\} \\ &\qquad - \left(\frac{\partial \xi_1}{\partial X_0}\right)^2\{\!\{\tilde{\epsilon}E_0^2\}\!\} \quad \text{at} \quad y = 0,\end{aligned} \tag{A 17}$$

with the solutions

$$\xi_2 = \Omega_E D^2 e^{2i(K'X_0 - \omega_0 T_0)} + \text{c.c.}, \tag{A 18}$$

$$\psi_2^{(2)} = -iE_0^{(2)} \frac{\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)} + \tilde{\epsilon}^{(1)}} y \frac{\partial D}{\partial X_1} e^{i(K'X_0 - \omega_0 T_0) - K'y} - E_0^{(2)} \frac{\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)} + \tilde{\epsilon}^{(1)}} (\Omega_E + K') D^2 e^{2i(K'X_0 - \omega_0 T_0) - 2K'y} + \text{c.c.}, \tag{A 19}$$

$$\psi_2^{(1)} = -iE_0^{(1)} \frac{\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)} + \tilde{\epsilon}^{(1)}} y \frac{\partial D}{\partial X_1} e^{i(K'X_0 - \omega_0 T_0) + K'y} + E_0^{(1)} \frac{\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)}}{\tilde{\epsilon}^{(2)} + \tilde{\epsilon}^{(1)}} (\Omega_E - K') D^2 e^{2i(K'X_0 - \omega_0 T_0) + 2K'y} + \text{c.c.}, \tag{A 20}$$

$$\phi_2^{(2)} = -\frac{1}{K'}\left[\frac{\partial D}{\partial T_1} + \frac{\omega_0}{K'}(1 + K'y) - \frac{\partial D}{\partial X_1}\right] e^{i(K'X_0 - \omega_0 T_0) - K'y} + \frac{i\omega_0}{K'} (\Omega_E + K') D^2 e^{2i(K'X_0 - \omega_0 T_0) - 2K'y} + \text{c.c.}, \tag{A 21}$$

$$\phi_2^{(1)} = \frac{1}{K'}\left[\frac{\partial D}{\partial T_1} + \frac{\omega_0}{K'}(1 - K'y) \frac{\partial D}{\partial X_1}\right] e^{i(K'X_0 - \omega_0 T_0) + K'y} - i\frac{\omega_0}{K'} (\Omega_E - K') D^2 e^{2i(K'K_0 - \omega_0 T_0) + 2K'y} + \text{c.c.}, \tag{A 22}$$

where c.c. denotes complex conjugate and Ω_E is given by (21). The above set of solutions is valid provided that the solvability condition given by (18) is satisfied.

For order ϵ^3 we have the following set of equations:

$$\frac{\partial^2 \psi_3^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \psi_3^{(2),(1)}}{\partial y^2} = -2\frac{\partial^2 \psi_2^{(2),(1)}}{\partial X_0 \partial X_1} - 2\frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_0 \partial X_2} - \frac{\partial^2 \psi_1^{(2),(1)}}{\partial X_1^2}, \tag{A 23}$$

$$\frac{\partial^2 \phi_3^{(2),(1)}}{\partial X_0^2} + \frac{\partial^2 \phi_3^{(2),(1)}}{\partial y^2} = -2\frac{\partial^2 \phi_2^{(2),(1)}}{\partial X_0 \partial X_1} - 2\frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0 \partial X_2} - \frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_1^2}, \tag{A 24}$$

$$\begin{aligned}\left\{\!\left|\frac{\partial \psi_3}{\partial X_0}\right|\!\right\} - \frac{\partial \xi_3}{\partial X_0}\{\!| E_0 |\!\} = {} & \frac{\partial \xi_2}{\partial X_1}\{\!| E_0 |\!\} + \frac{\partial \xi_1}{\partial X_2}\{\!| E_0 |\!\} - \xi_1 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\frac{\partial^2 \psi_1}{\partial y^2}\right|\!\right\} \\ & - \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\frac{\partial \psi_2}{\partial y}\right|\!\right\} - \frac{\partial \xi_2}{\partial X_0}\left\{\!\left|\frac{\partial \psi_1}{\partial y}\right|\!\right\} - \frac{\partial \xi_1}{\partial X_1}\left\{\!\left|\frac{\partial \psi_1}{\partial y}\right|\!\right\} \\ & - \xi_2\left\{\!\left|\frac{\partial^2 \psi_1}{\partial y \partial X_0}\right|\!\right\} - \tfrac{1}{2}\xi_1^2\left\{\!\left|\frac{\partial^3 \psi_1}{\partial y^2 \partial X_0}\right|\!\right\} - \xi_1\left\{\!\left|\frac{\partial^2 \psi_2}{\partial y \partial X_0}\right|\!\right\} \\ & - \xi_1\left\{\!\left|\frac{\partial^2 \psi_1}{\partial y \partial X_1}\right|\!\right\} - \left\{\!\left|\frac{\partial \psi_2}{\partial X_1}\right|\!\right\} - \left\{\!\left|\frac{\partial \psi_1}{\partial X_2}\right|\!\right\} \quad \text{at} \quad y = 0,\end{aligned} \tag{A 25}$$

$$\begin{aligned}\left\{\!\left|\tilde{\epsilon}\frac{\partial \psi_3}{\partial y}\right|\!\right\} = {} & \xi_1 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon}\frac{\partial^2 \psi_1}{\partial y \partial X_0}\right|\!\right\} + \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon}\frac{\partial \psi_2}{\partial X_0}\right|\!\right\} + \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon}\frac{\partial \psi_1}{\partial X_1}\right|\!\right\} \\ & + \left(\frac{\partial \xi_2}{\partial X_0} + \frac{\partial \xi_1}{\partial X_1}\right)\left\{\!\left|\tilde{\epsilon}\frac{\partial \psi_1}{\partial X_0}\right|\!\right\} - \xi_2\left\{\!\left|\tilde{\epsilon}\frac{\partial^2 \psi_1}{\partial y^2}\right|\!\right\} \\ & - \tfrac{1}{2}\xi_1^2\left\{\!\left|\tilde{\epsilon}\frac{\partial^3 \psi_1}{\partial y^3}\right|\!\right\} - \xi_1\left\{\!\left|\tilde{\epsilon}\frac{\partial^2 \psi_2}{\partial y^2}\right|\!\right\} \quad \text{at} \quad y = 0,\end{aligned} \tag{A 26}$$

$$
\begin{aligned}
\frac{\partial \zeta_3}{\partial T_0}-\frac{\partial \phi_3^{(2),(1)}}{\partial y}= & -\frac{\partial \xi_1}{\partial T_2}-\frac{\partial \xi_2}{\partial T_1}+\xi_1 \frac{\partial^2 \phi_2^{(2),(1)}}{\partial y^2}+\xi_2 \frac{\partial^2 \phi_1^{(2),(1)}}{\partial y^2}+\tfrac{1}{2} \xi_1^2 \frac{\partial^3 \phi_1^{(2),(1)}}{\partial y^3}-\frac{\partial \phi_1^{(2),(1)}}{\partial X_0} \frac{\partial \xi_2}{\partial X_0} \\
& -\frac{\partial \phi_2^{(2),(1)}}{\partial X_0} \frac{\partial \xi_1}{\partial X_0}-\frac{\partial \phi_1^{(2),(1)}}{\partial X_1} \frac{\partial \xi_1}{\partial X_0}-\frac{\partial \phi_1^{(2),(1)}}{\partial X_0} \frac{\partial \xi_1}{\partial X_1}-\xi_1 \frac{\partial^2 \phi_1^{(2),(1)}}{\partial X_0 \partial y} \frac{\partial \xi_1}{\partial X_0} \\
& \text{at} \quad y=0, \qquad \text{(A 27)}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial \phi_3^{(1)}}{\partial T_0}-\rho \frac{\partial \phi_3^{(2)}}{\partial T_0}+(1-\rho) \xi_3-\frac{\partial^2 \xi_3}{\partial X_0^2}+\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_3}{\partial y}\right|\!\right\} \\
& =-\frac{\partial \phi_1^{(1)}}{\partial T_2}-\frac{\partial \phi_2^{(1)}}{\partial T_1}+\rho \frac{\partial \phi_1^{(2)}}{\partial T_2}+\rho \frac{\partial \phi_2^{(2)}}{\partial T_1}+2 \frac{\partial^2 \xi_1}{\partial X_0 \partial X_2}+2 \frac{\partial^2 \xi_2}{\partial X_0 \partial X_1}+\frac{\partial^2 \xi_1}{\partial X_1^2} \\
& \quad-\frac{\partial \phi_1^{(1)}}{\partial X_0} \frac{\partial \phi_2^{(1)}}{\partial X_0}-\frac{\partial \phi_1^{(1)}}{\partial X_0} \frac{\partial \phi_1^{(1)}}{\partial X_0}-\xi_1 \frac{\partial \phi_1^{(1)}}{\partial X_0} \frac{\partial^2 \phi_1^{(1)}}{\partial X_0 \partial y}+\rho \frac{\partial \phi_1^{(2)}}{\partial X_0} \frac{\partial \phi_2^{(2)}}{\partial X_0}+\rho \frac{\partial \phi_1^{(2)}}{\partial X_0} \frac{\partial \phi_1^{(2)}}{\partial X_1} \\
& \quad+\rho \xi_1 \frac{\partial \phi_1^{(2)}}{\partial X_0} \frac{\partial^2 \phi_1^{(2)}}{\partial X_0 \partial y}-\frac{\partial \phi_1^{(1)}}{\partial y} \frac{\partial \phi_2^{(1)}}{\partial y}-\xi_1 \frac{\partial \phi_1^{(1)}}{\partial y} \frac{\partial^2 \phi_1^{(1)}}{\partial y^2}+\rho \frac{\partial \phi_1^{(2)}}{\partial y} \frac{\partial \phi_2^{(2)}}{\partial y}+\rho \xi_1 \frac{\partial \phi_1^{(2)}}{\partial y} \frac{\partial^2 \phi_1^{(2)}}{\partial y^2} \\
& \quad-\xi_2 \frac{\partial^2 \phi_1^{(1)}}{\partial T_0 \partial y}-\xi_1 \frac{\partial^2 \phi_2^{(1)}}{\partial T_0 \partial y}-\tfrac{1}{2} \xi_1^2 \frac{\partial^3 \phi_1^{(1)}}{\partial T_0 \partial y^2}+\rho \xi_2 \frac{\partial^2 \phi_1^{(2)}}{\partial T_0 \partial y}+\rho \xi_1 \frac{\partial^2 \phi_2^{(2)}}{\partial T_0 \partial y}+\tfrac{1}{2} \rho \xi_1^2 \frac{\partial^3 \phi_1^{(2)}}{\partial T_0 \partial y^2} \\
& \quad-\xi_1 \frac{\partial^2 \phi_1^{(1)}}{\partial T_1 \partial y}+\rho \xi_1 \frac{\partial^2 \phi_1^{(2)}}{\partial T_1 \partial y}-\frac{3}{2} \frac{\partial^2 \xi_1}{\partial X_0^2}\left(\frac{\partial \xi_1}{\partial X_0}\right)^2-\xi_1\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial X_0} \frac{\partial^2 \psi_1}{\partial X_0 \partial y}\right|\!\right\}-\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial X_0} \frac{\partial \psi_2}{\partial X_0}\right|\!\right\} \\
& \quad-\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial X_0} \frac{\partial \psi_1}{\partial X_1}\right|\!\right\}-\xi_2\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial^2 \psi_1}{\partial y^2}\right|\!\right\}-\tfrac{1}{2} \xi_1^2\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial^3 \psi_1}{\partial y^3}\right|\!\right\}-\xi_1\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial^2 \psi_2}{\partial y^2}\right|\!\right\} \\
& \quad+\xi_1\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial y} \frac{\partial^2 \psi_1}{\partial y^2}\right|\!\right\}+\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial y} \frac{\partial \psi_2}{\partial y}\right|\!\right\}+2 \xi_1 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial^2 \psi_1}{\partial X_0 \partial y}\right|\!\right\}+2 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_2}{\partial X_0}\right|\!\right\} \\
& \quad+2 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_1}{\partial X_1}\right|\!\right\}+2 \frac{\partial \xi_2}{\partial X_0}\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_1}{\partial X_0}\right|\!\right\}+2 \frac{\partial \xi_1}{\partial X_1}\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_1}{\partial X_0}\right|\!\right\}-2 \frac{\partial \xi_1}{\partial X_0}\left\{\!\left|\tilde{\epsilon} \frac{\partial \psi_1}{\partial X_0} \frac{\partial \psi_1}{\partial y}\right|\!\right\} \\
& \quad-\left[2 \frac{\partial \xi_1}{\partial X_0} \frac{\partial \xi_2}{\partial X_0}+2 \frac{\partial \xi_1}{\partial X_0} \frac{\partial \xi_1}{\partial X_1}\right]\left\{\!\left|\tilde{\epsilon} E_0^2\right|\!\right\}+2\left(\frac{\partial \xi_1}{\partial X_0}\right)^2\left\{\!\left|\tilde{\epsilon} E_0 \frac{\partial \psi_1}{\partial y}\right|\!\right\} \quad \text{at} \quad y=0. \qquad \text{(A 28)}
\end{aligned}
$$

The solutions of the above set of equations yield the solvability condition given by (19).

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